

Isotropic and Dominating Mixed Lizorkin–Triebel Spaces – a Comparison

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Abstract

We shall compare isotropic Lizorkin-Triebel spaces with their counterparts of dominating mixed smoothness.

1 Introduction

Let $t \in \mathbb{N}_0$ and $1 < p < \infty$. The standard isotropic Sobolev spaces is defined as

$$W_p^t(\mathbb{R}^d) = \left\{ f \in L_p(\mathbb{R}^d) : \|f|W_p^t(\mathbb{R}^d)\| = \sum_{|\bar{\alpha}|_1 \leq t} \|D^{\bar{\alpha}} f|L_p(\mathbb{R}^d)\| < \infty \right\}.$$

The Sobolev spaces of dominating mixed smoothness is given by

$$S_p^t W(\mathbb{R}^d) = \left\{ f \in L_p(\mathbb{R}^d) : \|f|S_p^t W(\mathbb{R}^d)\| = \sum_{|\bar{\alpha}|_\infty \leq t} \|D^{\bar{\alpha}} f|L_p(\mathbb{R}^d)\| < \infty \right\}.$$

Here $\bar{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\bar{\alpha}|_1 = \alpha_1 + \dots + \alpha_d$ and $|\bar{\alpha}|_\infty = \max_{i=1, \dots, d} |\alpha_i|$. Obviously we have the chain of continuous embeddings

$$W_p^{td}(\mathbb{R}^d) \hookrightarrow S_p^t W(\mathbb{R}^d) \hookrightarrow W_p^t(\mathbb{R}^d).$$

Also easy to see is the optimality of these embeddings in various directions. We will discuss this below. These two types of Sobolev spaces $W_p^t(\mathbb{R}^d)$ and $S_p^t W(\mathbb{R}^d)$ represent particular cases of corresponding scales of Bessel potential spaces (Sobolev spaces of fractional order t of smoothness), denoted by $H_p^t(\mathbb{R}^d)$ (isotropic smoothness) and $S_p^t H(\mathbb{R}^d)$ (dominating mixed smoothness). Let $t \in \mathbb{R}$ and $1 < p < \infty$. Then the space $H_p^t(\mathbb{R}^d)$ is the collection of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|H_p^t(\mathbb{R}^d)\| = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{t}{2}} \mathcal{F}f(\xi)](\cdot)|L_p(\mathbb{R}^d)\| < \infty,$$

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whereas $S_p^t H(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|S_p^t H(\mathbb{R}^d)\| = \left\| \mathcal{F}^{-1} \left[\prod_{i=1}^d (1 + \xi_i^2)^{\frac{t}{2}} \mathcal{F}f(\xi) \right] (\cdot) \right\|_{L_p(\mathbb{R}^d)} < \infty.$$

Here $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Indeed, if $t \in \mathbb{N}_0$ we have

$$W_p^t(\mathbb{R}^d) = H_p^t(\mathbb{R}^d) \quad \text{and} \quad S_p^t W(\mathbb{R}^d) = S_p^t H(\mathbb{R}^d)$$

in the sense of equivalent norms. In case $t > 0$ Schmeisser [15] stated that

$$H_p^{td}(\mathbb{R}^d) \hookrightarrow S_p^t H(\mathbb{R}^d) \hookrightarrow H_p^t(\mathbb{R}^d). \quad (1.1)$$

In this paper we shall give a proof of (1.1) and we shall show the optimality of these assertions in the following directions:

- Within all spaces $S_{p_0}^{t_0} H(\mathbb{R}^d)$ satisfying $S_{p_0}^{t_0} H(\mathbb{R}^d) \hookrightarrow H_p^t(\mathbb{R}^d)$ the class $S_p^t H(\mathbb{R}^d)$ is the largest one.
- Within all spaces $H_{p_0}^{t_0}(\mathbb{R}^d)$ satisfying $H_{p_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_p^t H(\mathbb{R}^d)$ the class $H_p^{td}(\mathbb{R}^d)$ is the largest one.
- Within all spaces $S_{p_0}^{t_0} H(\mathbb{R}^d)$ satisfying $H_2^{td}(\mathbb{R}^d) \hookrightarrow S_{p_0}^{t_0} H(\mathbb{R}^d)$ the class $S_p^t H(\mathbb{R}^d)$ is the smallest one.

In what follows we shall go one step further. Isotropic Sobolev spaces $H_p^t(\mathbb{R}^d)$ and Sobolev spaces of dominating mixed smoothness $S_p^t H(\mathbb{R}^d)$ of fractional order t are contained as special cases in the scales of isotropic Lizorkin-Triebel spaces $F_{p,q}^t(\mathbb{R}^d)$ and Lizorkin-Triebel spaces of dominating mixed smoothness $S_{p,q}^t F(\mathbb{R}^d)$. It is well-known that

$$H_p^t(\mathbb{R}^d) = F_{p,2}^t(\mathbb{R}^d) \quad \text{and} \quad S_p^t H(\mathbb{R}^d) = S_{p,2}^t F(\mathbb{R}^d), \quad 1 < p < \infty, \quad t \in \mathbb{R},$$

in the sense of equivalent norm, see [23, Theorem 2.5.6] and [16, Theorem 2.3.1]. In this paper we address the question under which conditions on t, p, q the embeddings

$$F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$$

hold true. In addition we shall discuss the optimality of these embeddings in various directions.

Nowadays isotropic Lizorkin-Triebel spaces represent a well accepted regularity notion in various fields of mathematics. Lizorkin-Triebel spaces of dominating mixed smoothness, in particular the scale $S_p^t H(\mathbb{R}^d)$, are of increasing importance in approximation theory and information-based complexity. As special cases, the scale $S_p^t H(\mathbb{R}^d) = S_{p,2}^t F(\mathbb{R}^d)$ contains the tensor products of the univariate spaces $H_p^t(\mathbb{R})$, and the scale $S_{p,p}^t F(\mathbb{R}^d)$ contains the tensor products of the univariate spaces $F_{p,p}^t(\mathbb{R})$, see [19, 20]. It is the main aim of this paper to give a detailed comparison of these different extensions of univariate Lizorkin-Triebel spaces into the multi-dimensional situation.

The paper is organized as follows. Section 2 is devoted to the definition and some basic properties of the function spaces under consideration. Our main results are stated in Section 3. Almost all proofs are concentrated in Section 4. In Subsection 4.1 we collect the required tools from Fourier analysis, especially some vector-valued Fourier multiplier assertions. The next Subsection 4.2 is devoted to complex interpolation. Dual spaces are discussed in Subsection 4.3. Finally, we collect families of test functions in Subsection 4.4.

Notation

As usual, \mathbb{N} denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, \mathbb{Z} the integers and \mathbb{R} the real numbers, \mathbb{C} refers to the complex numbers. For a real number a we put $a_+ := \max(a, 0)$. The letter d is always reserved for the underlying dimension in $\mathbb{R}^d, \mathbb{Z}^d$. We denote by $\langle x, y \rangle$ or $x \cdot y$ the usual Euclidean inner product in \mathbb{R}^d or \mathbb{C}^d . By $x \diamond y$ we mean

$$x \diamond y = (x_1 y_1, \dots, x_d y_d) \in \mathbb{R}^d.$$

If $\bar{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, then we put

$$|\bar{k}|_1 := k_1 + \dots + k_d \quad \text{and} \quad |\bar{k}|_\infty := \max_{j=1, \dots, d} |k_j|.$$

For $\bar{k} \in \mathbb{N}_0^d$ and $a > 0$ we write $a^{\bar{k}} := (a^{k_1}, \dots, a^{k_d})$. By C, C_1, C_2, \dots we denote positive constants which are independent of the main parameters involved but whose values may differ from line to line. The symbol $A \asymp B$ means that there exist positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. Let X and Y be two quasi-Banach spaces. Then $X \hookrightarrow Y$ indicates that the embedding is continuous. Let $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, be the space of all functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$\|f\|_{L_p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p} < \infty$$

with the usual modification if $p = \infty$. By $C_0^\infty(\mathbb{R}^d)$ the set of all compactly supported infinitely differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is denoted. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \mathbb{R}^d . The topological dual, the class of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$ (equipped with the weak topology). The Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ is given by

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^d.$$

The inverse transformation is denoted by \mathcal{F}^{-1} . We use both notations also for the transformations defined on $\mathcal{S}'(\mathbb{R}^d)$. Let $0 < p, q \leq \infty$. For a sequence of complex-valued functions $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ on \mathbb{R}^d , we put

$$\|f_{\bar{k}}\|_{L_p(\ell_q)} = \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |f_{\bar{k}}|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}.$$

2 Spaces of isotropic and dominating mixed smoothness

The isotropic spaces $F_{p,q}^t(\mathbb{R}^d)$ are invariant under rotations, the spaces $S_{p,q}^t F(\mathbb{R}^d)$ are not invariant under rotations. Both properties have been known for a long time and are well reflected by trace assertions on hyperplanes, see, e.g., Triebel [23, 2.7] (isotropic spaces) and Triebel [24], Vybíral [27], Vybíral and S. [29] (dominating mixed smoothness).

2.1 Isotropic Besov-Lizorkin-Triebel spaces

For us it will be convenient to introduce Lizorkin-Triebel and Besov spaces simultaneously. Let $\phi_0 \in \mathcal{S}(\mathbb{R}^d)$ be a non-negative function such that $\phi_0(x) = 1$ if $|x| \leq 1$ and $\phi_0(x) = 0$ if $|x| \geq \frac{3}{2}$. For $j \in \mathbb{N}$ we define

$$\phi_j(x) := \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^d.$$

Definition 2.1. Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$.

(i) The Besov space $B_{p,q}^t(\mathbb{R}^d)$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|B_{p,q}^t(\mathbb{R}^d)\|^\phi := \left(\sum_{j=0}^{\infty} 2^{j tq} \|\mathcal{F}^{-1}[\phi_j \mathcal{F}f](\cdot)\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}$$

is finite.

(ii) Let $p < \infty$. The Triebel-Lizorkin space $F_{p,q}^t(\mathbb{R}^d)$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|F_{p,q}^t(\mathbb{R}^d)\|^\phi := \left\| \left(\sum_{j=0}^{\infty} 2^{j tq} |\mathcal{F}^{-1}[\phi_j \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}$$

is finite.

Remark 2.2. Lizorkin-Triebel spaces are discussed in various monographs, let us refer, e.g., to [23], [25], [26] and [2]. They are quasi-Banach spaces (Banach spaces if $\min(p, q) \geq 1$) and they do not depend on the chosen generator ϕ_0 of the smooth dyadic decomposition (in the sense of equivalent quasi-norms). We call them isotropic because they are invariant under rotations. Characterizations in terms of differences can be found at various places, see, e.g., [23, 2.5], [25, 3.5] or [2, Sect. 28].

Many times we will work with the following equivalent quasi-norm. Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that for $x \in \mathbb{R}^d$

$$\psi_0(x) = 1 \quad \text{if} \quad \sup_{i=1,\dots,d} |x_i| \leq 1 \quad \text{and} \quad \psi_0(x) = 0 \quad \text{if} \quad \sup_{i=1,\dots,d} |x_i| \geq \frac{3}{2}.$$

For $j \in \mathbb{N}$, we define

$$\psi_j(x) := \psi_0(2^{-j}x) - \psi_0(2^{-j+1}x).$$

Then we have

$$\text{supp } \psi_j \subset \left\{ x : \sup_{i=1,\dots,d} |x_i| \leq 3 \cdot 2^{j-1} \right\} \setminus \left\{ x : \sup_{i=1,\dots,d} |x_i| \leq 2^{j-1} \right\}.$$

Proposition 2.3. Let $t \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then $F_{p,q}^t(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|F_{p,q}^t(\mathbb{R}^d)\|^\psi = \left\| \left(\sum_{j=0}^{\infty} 2^{j tq} |\mathcal{F}^{-1}[\psi_j \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}$$

is finite. The quasi-norms $\|f|F_{p,q}^t(\mathbb{R}^d)\|^\psi$ and $\|f|F_{p,q}^t(\mathbb{R}^d)\|^\phi$ are equivalent.

The equivalence of these quasi-norms has been proved in [23, Proposition 2.3.2] (in a much more general framework). Proposition 2.3 also holds true for Besov spaces (with the respective quasi-norms). From now on we will work with the ψ -norm and therefore we write $\|f|F_{p,q}^t(\mathbb{R}^d)\|$ instead of $\|f|F_{p,q}^t(\mathbb{R}^d)\|^\psi$.

2.2 Besov-Lizorkin-Triebel spaces of dominating mixed smoothness

Next we will give the definitions of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness. We start with a smooth dyadic decomposition on \mathbb{R} and afterwards we shall take its d -fold tensor product. More exactly, let $\varphi_0 \in C_0^\infty(\mathbb{R})$ satisfying $\varphi_0(\xi) = 1$ on $[-1, 1]$ and $\text{supp } \varphi \subset [-\frac{3}{2}, \frac{3}{2}]$. For $j \in \mathbb{N}$ we define

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \quad \xi \in \mathbb{R}. \quad (2.1)$$

Now we turn to tensor products. For $\bar{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ we put

$$\varphi_{\bar{k}}(x) := \varphi_{k_1}(x_1) \cdot \dots \cdot \varphi_{k_d}(x_d), \quad x \in \mathbb{R}^d. \quad (2.2)$$

This construction results in a smooth dyadic decomposition of unity $\{\varphi_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ on \mathbb{R}^d .

Definition 2.4. Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$.

(i) The Besov space of dominating mixed smoothness $S_{p,q}^t B(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|S_{p,q}^t B(\mathbb{R}^d)\| := \left(\sum_{\bar{k} \in \mathbb{N}_0^d} 2^{|\bar{k}|_1 t q} \|\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F}f](\cdot)|L_p(\mathbb{R}^d)\|^q \right)^{1/q}$$

is finite.

(ii) Let $0 < p < \infty$. The Lizorkin-Triebel space of dominating mixed smoothness $S_{p,q}^t F(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|S_{p,q}^t F(\mathbb{R}^d)\| := \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} 2^{|\bar{k}|_1 t q} |\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \quad (2.3)$$

is finite.

Remark 2.5. (i) For $d = 1$ we have $S_{p,q}^t A(\mathbb{R}) = A_{p,q}^t(\mathbb{R})$, $A \in \{B, F\}$.

(ii) Lizorkin-Triebel spaces of dominating mixed smoothness have a cross-quasi-norm, i.e., if $f_i \in F_{p,q}^t(\mathbb{R})$, $i = 1, \dots, d$, then it follows

$$f(x) = \prod_{i=1}^d f_i(x_i) \in S_{p,q}^t F(\mathbb{R}^d) \quad \text{and} \quad \|f|S_{p,q}^t F(\mathbb{R}^d)\| = \prod_{i=1}^d \|f_i|F_{p,q}^t(\mathbb{R})\|.$$

Of certain use for us will be the following Nikol'skij representation for Lizorkin-Triebel spaces of dominating mixed smoothness.

Proposition 2.6. Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $t \in \mathbb{R}$. Let further $\{\varphi_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ be the above system. Then the space $S_{p,q}^t F(\mathbb{R}^d)$ is a collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that there exists a sequence $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \subset L_p(\mathbb{R}^d)$ satisfying

$$f = \sum_{\bar{k} \in \mathbb{N}_0^d} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f_{\bar{k}} \quad \text{in } \mathcal{S}'(\mathbb{R}^d) \quad \text{and} \quad \|2^{t|\bar{k}|_1} f_{\bar{k}}|L_p(\ell_q)\| < \infty. \quad (2.4)$$

The norm

$$\|f|S_{p,q}^t F(\mathbb{R}^d)\|^* := \inf \|2^{t|\bar{k}|_1} f_{\bar{k}}|L_p(\ell_q)\|$$

is equivalent to the norm in (2.3). Here the infimum is taken over all admissible representations in (2.4).

Proof. *Step 1.* Let $\{\varphi_j\}_{j=0}^\infty$ be the system given in (2.1). We put

$$\tilde{\varphi}_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j \in \mathbb{N}_0, \quad (2.5)$$

with $\varphi_{-1} \equiv 0$. If $\bar{k} \in \mathbb{N}_0^d$ we define $\tilde{\varphi}_{\bar{k}} := \tilde{\varphi}_{k_1} \otimes \cdots \otimes \tilde{\varphi}_{k_d}$. For $f \in S_{p,q}^t F(\mathbb{R}^d)$ we choose $f_{\bar{k}} = \mathcal{F}^{-1} \tilde{\varphi}_{\bar{k}} \mathcal{F} f$. It follows from $\sum_{\bar{k} \in \mathbb{Z}^d} \varphi_{\bar{k}} = 1$ for all $x \in \mathbb{R}^d$ and $\tilde{\varphi}_j(x) = 1$ if $x \in \text{supp } \varphi_j$ that

$$\sum_{\bar{k} \in \mathbb{N}_0^d} \mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} f_{\bar{k}}] = \sum_{\bar{k} \in \mathbb{N}_0^d} \mathcal{F}^{-1} [\varphi_{\bar{k}} \tilde{\varphi}_{\bar{k}} \mathcal{F} f] = f.$$

Hence

$$\begin{aligned} \|f|S_{p,q}^t F(\mathbb{R}^d)\|^* &\leq \|2^{t|\bar{k}|_1} f_{\bar{k}}|L_p(\ell_q)\| \\ &= \|2^{t|\bar{k}|_1} \mathcal{F}^{-1} \tilde{\varphi}_{\bar{k}} \mathcal{F} f|L_p(\ell_q)\| \leq 3^d \|2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f|L_p(\ell_q)\|. \end{aligned}$$

Step 2. Assume that f can be represented as in (2.4). We put $\varphi_{\bar{k}} \equiv 0$ if $\min_{i=1,\dots,d} k_i < 0$. Then we obtain

$$\mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f = \mathcal{F}^{-1} \left(\varphi_{\bar{k}} \sum_{\bar{\ell} \in \{-1,0,1\}^d} \varphi_{\bar{k}+\bar{\ell}} \mathcal{F} f_{\bar{k}+\bar{\ell}} \right).$$

Applying Lemma 4.6 we get

$$\begin{aligned} \|2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f|L_p(\ell_q)\| &\leq c_1 \left\| 2^{t|\bar{k}|_1} \mathcal{F}^{-1} \left(\sum_{\bar{\ell} \in \{-1,0,1\}^d} \varphi_{\bar{k}+\bar{\ell}} \mathcal{F} f_{\bar{k}+\bar{\ell}} \right) \right\|_{L_p(\ell_q)} \\ &\leq c_2 \|2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f_{\bar{k}}|L_p(\ell_q)\|. \end{aligned}$$

To continue we split $\sum_{\bar{k}}$ into several parts. Observe that

$$\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} f_{\bar{k}}]|^q = \sum_{e \subset \{1,\dots,d\}} \sum_{\substack{k_i \geq 1, i \in e \\ k_j = 0, j \notin e}} |2^{t|\bar{k}|_1} \mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} f_{\bar{k}}]|^q. \quad (2.6)$$

Proposition 4.3 can be applied to each subsum. This yields

$$\left\| 2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f_{\bar{k}} \right\|_{L_p(\ell_q)} \leq c_3 \|2^{t|\bar{k}|_1} f_{\bar{k}}\|_{L_p(\ell_q)}$$

with a constant c_3 independent of f . The proof is complete. ■

3 The main results

As mentioned in the Introduction we will split our considerations into two cases:

- $S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d);$
- $F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d).$

3.1 The embedding of dominating mixed spaces into isotropic spaces

The first of our main results reads as follows.

Theorem 3.1. *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$ and $t \in \mathbb{R}$. Then we have*

$$S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$$

if one of the following conditions is satisfied:

- $t > 0$;
- $t = 0$, $1 < p < \infty$ and $0 < q \leq 2$;
- $t = 0$, $0 < p \leq 1$ and $0 < q < 2$;

Remark 3.2. We recall that $S_{p,2}^0 F(\mathbb{R}^d) = F_{p,2}^0(\mathbb{R}^d) = L_p(\mathbb{R}^d)$, $1 < p < \infty$, in the sense of equivalent norms. This is a consequence of certain Littlewood-Paley assertions, see Nikol'skij [14, 1.5.6]. This identity does not extend to $p = 1$. Here we conjecture

$$S_{1,2}^0 F(\mathbb{R}^d) \hookrightarrow F_{1,2}^0(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d).$$

Theorem 3.3. *Let $d \geq 2$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $t \in \mathbb{R}$. Then*

$$S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$$

if and only if either $t > 0$ or $t = 0$ and $1 \leq q \leq 2$.

In addition we have the following.

Proposition 3.4. *Let $d \geq 2$ and $t < 0$.*

- (i) *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then $F_{p,q}^t(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$.*
- (ii) *If $0 < p < 1$, $0 < q \leq \infty$, then $F_{p,q}^t(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$ are not comparable.*

We summarize the relation between $F_{p,q}^t(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$ ($1 \leq q \leq \infty$) in the following figure.

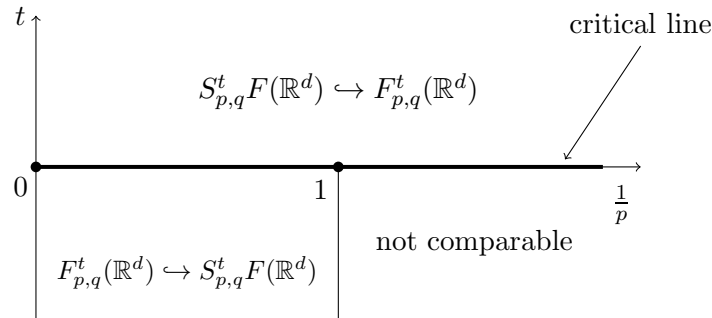


Figure 1. Comparison of $S_{p,q}^t F(\mathbb{R}^d)$ and $F_{p,q}^t(\mathbb{R}^d)$

These above embeddings are optimal in the following sense.

Theorem 3.5. *Let $d \geq 2$, $0 < p_0, p < \infty$, $0 < q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let p, q and t be fixed. Within all spaces $S_{p_0, q_0}^{t_0} F(\mathbb{R}^d)$ satisfying*

$$S_{p_0, q_0}^{t_0} F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$$

the class $S_{p,q}^t F(\mathbb{R}^d)$ is the largest.

Remark 3.6. Note that, within all spaces $F_{p_0, q_0}^{t_0}(\mathbb{R}^d)$ satisfying $S_{p, q}^t F(\mathbb{R}^d) \hookrightarrow F_{p_0, q_0}^{t_0}(\mathbb{R}^d)$ the class $F_{p, q}^t(\mathbb{R}^d)$ is not the smallest one. There is no smallest space with this respect. We may follow the arguments we have used in the context of the analogous problem for Besov spaces, see [13]. From Theorem 3.1 we have $S_{1, 2}^2 F(\mathbb{R}^d) \hookrightarrow F_{1, 2}^2(\mathbb{R}^d)$. On the other hand, a Sobolev-type embedding and Theorem 3.1 imply

$$S_{1, 2}^2 F(\mathbb{R}^d) \hookrightarrow S_{2, 2}^{3/2} F(\mathbb{R}^d) \hookrightarrow F_{2, 2}^{3/2}(\mathbb{R}^d).$$

However, for $d \geq 2$ the spaces $F_{1, 2}^2(\mathbb{R}^d)$ and $F_{2, 2}^{3/2}(\mathbb{R}^d)$ are not comparable.

3.2 The embedding of isotropic spaces into dominating mixed spaces

Theorem 3.7. *Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$ and $t \in \mathbb{R}$. Then we have*

$$F_{p, q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p, q}^t F(\mathbb{R}^d)$$

if one of the following conditions is satisfied:

- $t > \left(\frac{1}{\min(p, q)} - 1\right)_+$ and $0 < q < \infty$;
- $t = 0$, $1 < p < \infty$ and $2 \leq q \leq \infty$.

Remark 3.8. The proof below is a bit more general than stated in Theorem 3.7. Let $q = \infty$. Then we shall prove that $F_{p, \infty}^{td}(\mathbb{R}^d) \hookrightarrow S_{p, \infty}^t F(\mathbb{R}^d)$ if either $p > 1$ and $t > 0$ or $0 < p \leq 1$ and $t > 1/p$. For further comments, see Remark 4.13.

By using a similar argument as in proof of Theorem 3.3 we conclude the following.

Theorem 3.9. *Let $d \geq 2$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $t \in \mathbb{R}$. Then*

$$F_{p, q}^{td} \hookrightarrow S_{p, q}^t F(\mathbb{R}^d)$$

if and only if either $t > 0$ or $t = 0$ and $q \geq 2$.

In addition we have the following supplement.

Proposition 3.10. *Let $d \geq 2$.*

- (i) *Let $0 < p < 1$, $0 < q \leq \infty$ and $0 < t \leq \frac{1}{p} - 1$. Then $S_{p, q}^t F(\mathbb{R}^d)$ and $F_{p, q}^{td}(\mathbb{R}^d)$ are not comparable.*
- (ii) *Let $0 < p < \infty$, $0 < q \leq \infty$ and $t < 0$. Then $S_{p, q}^t F(\mathbb{R}^d) \hookrightarrow F_{p, q}^{td}(\mathbb{R}^d)$ follows.*

We summarize the relation between $F_{p, q}^{td}(\mathbb{R}^d)$ and $S_{p, q}^t F(\mathbb{R}^d)$ ($1 \leq q \leq \infty$) in the following figure.

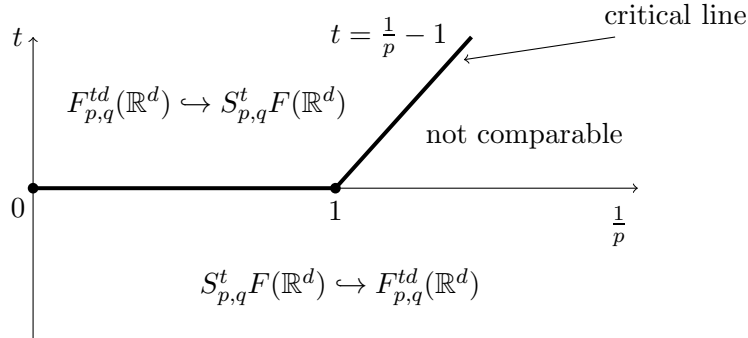


Figure 2. Comparison of $S_{p, q}^t F(\mathbb{R}^d)$ and $F_{p, q}^{td}(\mathbb{R}^d)$

These embeddings are optimal in the following sense.

Theorem 3.11. *Let $d \geq 2$, $0 < p_0, p < \infty$, $0 < q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let p, q and t be fixed. Within all spaces $F_{p_0, q_0}^{t_0}(\mathbb{R}^d)$ satisfying*

$$F_{p_0, q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_{p, q}^t F(\mathbb{R}^d)$$

the class $F_{p, q}^{td}(\mathbb{R}^d)$ is the largest.

Theorem 3.12. *Let $d \geq 2$, $0 < p_0, p < \infty$, $0 < q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let p, q and t be fixed. Within all spaces $S_{p_0, q_0}^{t_0} F(\mathbb{R}^d)$ satisfying*

$$F_{p, q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p_0, q_0}^{t_0} F(\mathbb{R}^d)$$

the class $S_{p, q}^t F(\mathbb{R}^d)$ is the smallest.

4 Proofs of the main results

To prove our main results we will apply essentially four different tools: vector-valued Fourier multipliers; complex interpolation; assertions on dual spaces and some test functions. In what follows we collect what is needed.

4.1 Tools from Fourier analysis

In this section we will collect the required tools from Fourier analysis.

For a locally integrable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ we denote by $Mf(x)$ the Hardy-Littlewood maximal function defined by

$$(Mf)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^d, \quad (4.1)$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes containing x . A vector-valued generalization of the classical Hardy-Littlewood maximal inequality is due to Fefferman and Stein [5].

Theorem 4.1. For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $C > 0$, such that

$$\|Mf_{\vec{k}}\|_{L_p(\ell_q)} \leq C \|f_{\vec{k}}\|_{L_p(\ell_q)}$$

holds for all sequences $\{f_{\vec{k}}(x)\}_{\vec{k} \in \mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on \mathbb{R}^d .

We require a direction-wise version of (4.1)

$$(M_i f)(x) = \sup_{s > 0} \frac{1}{2s} \int_{x_i - s}^{x_i + s} |f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d)| dt, \quad i = 1, \dots, d.$$

The following version of the Fefferman-Stein maximal inequality is due to Bagby [1], see also Stöckert [21].

Theorem 4.2. For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $C > 0$, such that for any $i = 1, \dots, d$

$$\|M_i f_{\vec{k}}\|_{L_p(\ell_q)} \leq C \|f_{\vec{k}}\|_{L_p(\ell_q)}$$

holds for all sequences $\{f_{\vec{k}}(x)\}_{\vec{k} \in \mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on \mathbb{R}^d .

Iterative application of this theorem yields a similar boundedness property for the operator $\mathcal{M} = M_d \circ \dots \circ M_1$.

The following proposition will be a consequence of Theorem 4.2. In its proof we will follow the arguments in the isotropic case, see [31].

Proposition 4.3. *Suppose $1 < p < \infty$, $1 \leq q \leq \infty$ and let $\phi(x) \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a constant C such that*

$$\|\mathcal{F}^{-1}[\phi(2^{-\bar{k}}\xi)\mathcal{F}f_{\bar{k}}(\xi)](\cdot)|L_p(\ell_q)\| \leq C\|f_{\bar{k}}(x)|L_p(\ell_q)\|$$

for all $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \in L_p(\ell_q)$.

Proof. *Step 1.* The case $1 < q \leq \infty$. Recall the notation $2^{\bar{k}} := (2^{k_1}, \dots, 2^{k_d})$. Observe that for $\bar{k} \in \mathbb{N}_0^d$ we have

$$\mathcal{F}^{-1}[\phi(2^{-\bar{k}} \diamond \cdot)\mathcal{F}f_{\bar{k}}(\cdot)](x) = (2\pi)^{-\frac{d}{2}} 2^{|\bar{k}|_1} \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(2^{\bar{k}} \diamond y) f_{\bar{k}}(x - y) dy. \quad (4.2)$$

Let $\alpha > 1$. The assumption $\phi \in \mathcal{S}(\mathbb{R}^d)$ implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(2^{\bar{k}} \diamond y) f_{\bar{k}}(x - y) dy \right| \\ & \leq \sup_{y \in \mathbb{R}^d} \left\{ \left(\prod_{i=1}^d (1 + |2^{k_i} y_i|^2)^{\frac{\alpha}{2}} \right) |(\mathcal{F}^{-1}\phi)(2^{\bar{k}} \diamond y)| \right\} \int_{\mathbb{R}^d} \left(\prod_{i=1}^d (1 + |2^{k_i} y_i|^2)^{-\frac{\alpha}{2}} \right) |f_{\bar{k}}(x - y)| dy \\ & \leq c_1 \int_{\mathbb{R}^d} \left(\prod_{i=1}^d (1 + |2^{k_i} y_i|^2)^{-\frac{\alpha}{2}} \right) |f_{\bar{k}}(x - y)| dy \end{aligned} \quad (4.3)$$

with a constant c_1 depending on ϕ , but not on \bar{k} and $f_{\bar{k}}$. For $\bar{\ell} \in \mathbb{Z}^d$ and $\bar{k} \in \mathbb{N}_0^d$ we put

$$P(\bar{k}, \bar{\ell}) := \{x \in \mathbb{R}^d : 2^{-k_i} 2^{\ell_i} \leq |x_i| < 2^{-k_i} 2^{\ell_i+1}, \quad i = 1, \dots, d\}.$$

Observe

$$\mathbb{R}^d = \bigcup_{\bar{\ell} \in \mathbb{Z}^d} P(\bar{k}, \bar{\ell}) \quad \text{for all } \bar{k} \in \mathbb{N}_0^d.$$

Then we obtain from (4.3)

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(2^{\bar{k}} \diamond y) f_{\bar{k}}(x - y) dy \right| \\ & \leq c_1 \sum_{\bar{\ell} \in \mathbb{Z}^d} \left(\sup_{y \in P(\bar{k}, \bar{\ell})} \prod_{i=1}^d (1 + |2^{k_i} y_i|^2)^{-\frac{\alpha}{2}} \right) \int_{P(\bar{k}, \bar{\ell})} |f_{\bar{k}}(x - y)| dy. \end{aligned} \quad (4.4)$$

By applying \mathcal{M} to the integral on the right-hand side of (4.4) we derive

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(2^{\bar{k}} \diamond y) f_{\bar{k}}(x - y) dy \right| & \leq c_1 (\mathcal{M}f_{\bar{k}})(x) \sum_{\bar{\ell} \in \mathbb{Z}^d} 2^{-|\bar{k}|_1} \sup_{y \in P(\bar{k}, \bar{\ell})} \prod_{i=1}^d \frac{2^{\ell_i}}{(1 + |2^{k_i} y_i|^2)^{\frac{\alpha}{2}}} \\ & \leq c_1 2^{-|\bar{k}|_1} (\mathcal{M}f_{\bar{k}})(x) \sum_{\bar{\ell} \in \mathbb{Z}^d} \prod_{i=1}^d \frac{2^{\ell_i}}{(1 + 2^{\ell_i})^\alpha} \\ & \leq c_2 2^{-|\bar{k}|_1} (\mathcal{M}f_{\bar{k}})(x). \end{aligned}$$

Inserting this into (4.2) we arrive at

$$|\mathcal{F}^{-1}[\phi(2^{-\bar{k}} \diamond \cdot) \mathcal{F}f_{\bar{k}}(\cdot)](x)| \leq c_3 (\mathcal{M}f_{\bar{k}})(x)$$

with some c_3 independent of \bar{k} and $f_{\bar{k}}$. Now the desired estimate follows from Theorem 4.2.

Step 2. The case $q = 1$. From Step 1 we derive that the linear operator

$$T : \{f_{\bar{k}}\}_{\bar{k}} \rightarrow \{\mathcal{F}^{-1}[\phi(2^{-\bar{k}} \diamond y) \mathcal{F}f_{\bar{k}}(y)]\}_{\bar{k}}$$

is bounded from $L_{p'}(\ell_\infty)$ into itself. By a duality argument we conclude that the adjoint operator T' of T is bounded from $[L_{p'}(\ell_\infty)]'$ into itself. That is

$$\|\mathcal{F}^{-1}[\bar{\phi}(2^{-\bar{k}} \diamond y) \mathcal{F}f_{\bar{k}}(y)](\cdot) \mid [L_{p'}(\ell_\infty)]'\| \leq C \|f_{\bar{k}} \mid [L_{p'}(\ell_\infty)]'\|$$

for all $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \in [L_{p'}(\ell_\infty)]'$. Of course the same inequality follows with $\bar{\phi}$ replaced by ϕ . The canonical embedding

$$L_p(\ell_1) \rightarrow [L_{p'}(\ell_\infty)]'$$

is a linear isometry, see, e.g., [30, Satz III.3.1]. We put

$$\mathcal{A} := \left\{ \{f_{\bar{k}}(x)\}_{\bar{k} \in \mathbb{N}_0^d} : f_{\bar{k}}(x) \in \mathcal{S}(\mathbb{R}^d), \right. \\ \left. f_{\bar{k}}(x) \equiv 0 \text{ for all but a finite number of } \bar{k} \right\}.$$

It is obvious that

$$\{\mathcal{F}^{-1}[\phi(2^{-\bar{k}} \diamond y) \mathcal{F}f_{\bar{k}}(y)]\}_{\bar{k} \in \mathbb{N}_0^d} \in \mathcal{A} \subset L_p(\ell_1)$$

if $\{f_{\bar{k}}(x)\}_{\bar{k} \in \mathbb{N}_0^d} \in \mathcal{A}$. Because \mathcal{A} is dense in $L_p(\ell_1)$ we conclude that

$$\|\mathcal{F}^{-1}[\phi(2^{-\bar{k}} \diamond y) \mathcal{F}f_{\bar{k}}(y)](\cdot) \mid L_p(\ell_1)\| \leq C \|f_{\bar{k}} \mid L_p(\ell_1)\|$$

holds for all $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \in L_p(\ell_1)$. The proof is complete. \blacksquare

Definition 4.4. Let $0 < p, q \leq \infty$. Let $\Omega = \{\Omega_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ be a sequence of compact subsets in \mathbb{R}^d . Then we define

$$L_p^\Omega(\ell_q) = \left\{ \{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} : f_{\bar{k}} \in \mathcal{S}'(\mathbb{R}^d), \text{ supp } \mathcal{F}f_{\bar{k}} \subset \Omega_{\bar{k}} \text{ if } \bar{k} \in \mathbb{N}_0^d, \|f_{\bar{k}}\|_{L_p(\ell_q)} < \infty \right\}.$$

We specify the sequence of compact subsets of \mathbb{R}^d by choosing

$$\Omega_{\bar{k}} := \{x \in \mathbb{R}^d : |x_{k_i}| \leq a_{k_i}, i = 1, \dots, d\}, \quad (4.5)$$

with $a_{\bar{k}} = (a_{k_1}, \dots, a_{k_d})$, $\bar{k} \in \mathbb{N}_0^d$, $a_{k_i} > 0$, $i = 1, \dots, d$. The following Proposition has been proved in [16, Theorem 1.10.2] for $d = 2$. A proof for general d can be found in [7, Proposition 2.3.4].

Proposition 4.5. Let $0 < p < \infty$, $0 < q \leq \infty$ and $\Omega = \{\Omega_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ be the sequence given in (4.5). Let $0 < s < \min(p, q)$. Then there exists a positive constant C , independent of the sequence $\{a_{\bar{k}}\}_{\bar{k}}$, such that

$$\left\| \sup_{z \in \mathbb{R}^d} \frac{|f_{\bar{k}}(\cdot - z)|}{\prod_{i=1}^d (1 + |a_{k_i} z_i|^{\frac{1}{s}})} \mid L_p(\ell_q) \right\| \leq C \|f_{\bar{k}} \mid L_p(\ell_q)\|$$

holds for all systems $\{f_{\bar{k}}\} \in L_p^\Omega(\ell_q)$.

Next we recall a Fourier multiplier assertion for the spaces $L_p^\Omega(\ell_q)$. We refer to [16, Theorem 1.10.3], see also [28, Theorem 1.12] or [7, Proposition 2.3.5].

Lemma 4.6. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\Omega = \{\Omega_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ be a sequence of compact subsets of \mathbb{R}^d given in (4.5). Let $r > \frac{1}{\min(p,q)} + \frac{1}{2}$. Then there exists a constant C , independent of the sequence $\{a_{\bar{k}}\}_{\bar{k}}$, such that*

$$\|\mathcal{F}^{-1} M_{\bar{k}} \mathcal{F} f_{\bar{k}}\|_{L_p(\ell_q)} \leq C \sup_{\bar{\ell} \in \mathbb{N}_0^d} \|M_{\bar{\ell}}(a_{\bar{\ell}}) S_2^r H(\mathbb{R}^d)\| \cdot \|f_{\bar{k}}\|_{L_p(\ell_q)}$$

holds for all systems $\{f_{\bar{k}}\}_{\bar{k}} \in L_p^\Omega(\ell_q)$ and all systems $\{M_{\bar{k}}\}_{\bar{k}} \in S_2^r H(\mathbb{R}^d)$.

4.2 Complex interpolation

For the basics of Calderón's complex interpolation method we refer to the monographs [3, 11, 22]. It is well-known that this complex interpolation method can be extended to a special class of quasi-Banach spaces, called analytically convex, see [10]. Note that any Banach space is analytically convex. The following proposition was well-known in the classical context of Banach spaces, see [11, Theorem 2.1.6], [3, Theorem 4.1.2] or [22, Theorem 1.10.3.1]. The extension to quasi-Banach spaces can be found in Kalton, Mayboroda and Mitrea [10].

Proposition 4.7. *Let $0 < \Theta < 1$. Let (X_1, Y_1) and (X_2, Y_2) be two compatible couples of quasi-Banach spaces. In addition, let $X_1 + Y_1$, $X_2 + Y_2$ be analytically convex. If T is in $\mathcal{L}(X_1, X_2)$ and in $\mathcal{L}(Y_1, Y_2)$, then the restriction of T to $[X_1, Y_1]_\Theta$ is in $\mathcal{L}([X_1, Y_1]_\Theta, [X_2, Y_2]_\Theta)$ for every Θ . Moreover,*

$$\|T : [X_1, Y_1]_\Theta \rightarrow [X_2, Y_2]_\Theta\| \leq \|T : X_1 \rightarrow X_2\|^{1-\Theta} \|T : Y_1 \rightarrow Y_2\|^\Theta.$$

Complex interpolation of isotropic Lizorkin-Triebel spaces has been studied, e.g., in [22, 23, 6, 10]. For the case of dominating mixed smoothness one can find the proof for associated sequence spaces in [28, Theorem 4.6]. However, these results can be shifted to the level of function spaces by some wavelet isomorphisms, see [28, Theorem 2.12].

Proposition 4.8. *Let $t_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$, $i = 1, 2$, and $\min(q_1, q_2) < \infty$. Let $0 < \Theta < 1$. If t_0, p_0 and q_0 are given by*

$$\frac{1}{p_0} = \frac{1-\Theta}{p_1} + \frac{\Theta}{p_2}, \quad \frac{1}{q_0} = \frac{1-\Theta}{q_1} + \frac{\Theta}{q_2}, \quad t_0 = (1-\Theta)t_1 + \Theta t_2.$$

Then

$$F_{p_0, q_0}^{t_0}(\mathbb{R}^d) = [F_{p_1, q_1}^{t_1}(\mathbb{R}^d), F_{p_2, q_2}^{t_2}(\mathbb{R}^d)]_\Theta$$

and

$$S_{p_0, q_0}^{t_0} F(\mathbb{R}^d) = [S_{p_1, q_1}^{t_1} F(\mathbb{R}^d), S_{p_2, q_2}^{t_2} F(\mathbb{R}^d)]_\Theta.$$

4.3 Dual spaces

Next we will recall some results about the dual spaces of $F_{p,q}^t(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$. Note that $\mathcal{S}(\mathbb{R}^d)$ is dense either in $F_{p,q}^t(\mathbb{R}^d)$ or in $S_{p,q}^t F(\mathbb{R}^d)$ if and only if $\max(p, q) < \infty$. By $\dot{F}_{p,q}^t(\mathbb{R}^d)$ we denote the closure of $\mathcal{S}(\mathbb{R}^d)$ in $F_{p,q}^t(\mathbb{R}^d)$ and by $\dot{S}_{p,q}^t F(\mathbb{R}^d)$ the closure of $\mathcal{S}(\mathbb{R}^d)$ in $S_{p,q}^t F(\mathbb{R}^d)$. The dual

space of $F_{p,q}^t(\mathbb{R}^d)$ must be understood in the following sense: $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the dual space $[F_{p,q}^t(\mathbb{R}^d)]'$ of $F_{p,q}^t(\mathbb{R}^d)$ if and only if there exists a positive constant C such that

$$|f(\varphi)| \leq C \|\varphi\|_{F_{p,q}^t(\mathbb{R}^d)} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Similarly for the space $S_{p,q}^t F(\mathbb{R}^d)$. For $1 < p < \infty$ the conjugate exponent p' is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. If $0 < p \leq 1$ we put $p' := \infty$ and if $p = \infty$ we put $p' := 1$. Let c_0 be the space of all sequences converging to zero. Let $L_p(c_0)$ denote the space of all sequences $\{\psi_{\bar{k}}\}_{\bar{k}}$ of measurable functions such that

$$\lim_{|\bar{k}|_1 \rightarrow \infty} |\psi_{\bar{k}}(x)| = 0 \quad \text{a.e.}$$

equipped with the norm

$$\|\psi_{\bar{k}}\|_{L_p(c_0)} := \left\| \sup_{\bar{k} \in \mathbb{N}_0^d} |\psi_{\bar{k}}(\cdot)| \right\|_{L_p(\mathbb{R}^d)}.$$

The following lemma is well-known, see, e.g., [23, Proposition 2.11.1] and [4, Theorems 8.18.2, 8.20.3].

Lemma 4.9. (i) *Let $1 \leq p < \infty$ and $0 < q < \infty$. Then $g \in (L_p(\ell_q))'$ if and only if it can be represented uniquely as*

$$g(f) = \sum_{\bar{k} \in \mathbb{N}_0^d} \int_{\mathbb{R}^d} g_{\bar{k}}(x) f_{\bar{k}}(x) dx$$

for every $f = \{f_{\bar{k}}(x)\}_{\bar{k} \in \mathbb{N}_0^d} \in L_p(\ell_q)$, where

$$g = \{g_{\bar{k}}(x)\}_{\bar{k} \in \mathbb{N}_0^d} \in L_{p'}(\ell_{q'}) \quad \text{and} \quad \|g\| = \|g_{\bar{k}}\|_{L_{p'}(\ell_{q'})}.$$

(ii) *Let $1 < p < \infty$. Then we have*

$$(L_p(c_0))' = L_{p'}(\ell_1).$$

Proposition 4.10. *Let $t \in \mathbb{R}$.*

(i) *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then*

$$[\hat{F}_{p,q}^t(\mathbb{R}^d)]' = F_{p',q'}^{-t}(\mathbb{R}^d) \quad \text{and} \quad [\hat{S}_{p,q}^t F(\mathbb{R}^d)]' = S_{p',q'}^{-t} F(\mathbb{R}^d).$$

(ii) *If $0 < p < 1$ and $0 < q \leq \infty$ then*

$$[\hat{F}_{p,q}^t(\mathbb{R}^d)]' = B_{\infty,\infty}^{-t+d(\frac{1}{p}-1)}(\mathbb{R}^d) \quad \text{and} \quad [\hat{S}_{p,q}^t F(\mathbb{R}^d)]' = S_{\infty,\infty}^{-t+\frac{1}{p}-1} B(\mathbb{R}^d).$$

Proof. The proof in the isotropic setting can be found in [23, Section 2.11] and [12]. Duality of spaces of dominating mixed smoothness has been considered in [7, Section 5.5]. But there only partial results with respect to sequence spaces associated to Lizorkin-Triebel spaces of dominating mixed smoothness can be found.

Step 1. The case $1 < p < \infty$ and $1 \leq q < \infty$.

Substep 1.1. We shall prove that $S_{p',q'}^{-t} F(\mathbb{R}^d) \hookrightarrow (S_{p,q}^t F(\mathbb{R}^d))'$. Let $f \in S_{p',q'}^{-t} F(\mathbb{R}^d)$. Since $1 < p' < \infty$ and $1 < q' \leq \infty$ we can find $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ such that

$$f = \sum_{\bar{k} \in \mathbb{N}_0^d} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f_{\bar{k}} \quad \text{in } \mathcal{S}'(\mathbb{R}^d) \quad \text{and} \quad \|2^{-|\bar{k}|_1 t} f_{\bar{k}}\|_{L_{p'}(\ell_{q'})} \leq 2 \|f\|_{S_{p',q'}^{-t} F(\mathbb{R}^d)}^*.$$

With $\varrho \in \mathcal{S}(\mathbb{R}^d)$ we conclude

$$\begin{aligned} |f(\varrho)| &= \left| \sum_{\bar{k} \in \mathbb{N}_0^d} (\mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f_{\bar{k}})(\varrho) \right| = \left| \sum_{\bar{k} \in \mathbb{N}_0^d} f_{\bar{k}}(\mathcal{F} \varphi_{\bar{k}} \mathcal{F}^{-1} \varrho) \right| \\ &\leq c_1 \|2^{-|\bar{k}|_1 t} f_{\bar{k}}\|_{L_{p'}(\ell_{q'})} \cdot \|2^{|\bar{k}|_1 t} \mathcal{F} \varphi_{\bar{k}} \mathcal{F}^{-1} \varrho\|_{L_p(\ell_q)} \\ &\leq c_2 \|f\|_{S_{p',q'}^{-t} F(\mathbb{R}^d)}^* \cdot \|\varrho\|_{S_{p,q}^t F(\mathbb{R}^d)}. \end{aligned}$$

Substep 1.2. Now we prove the reverse direction. Here we assume that the underlying decomposition of unity, see (2.1) and (2.2), is generated by an even function φ_0 . Then all elements of the sequence $\{\varphi_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ are even functions as well. Let $f \in S_{p,q}^t F(\mathbb{R}^d)$. Then the operator

$$f \rightarrow \{2^{|\bar{k}|_1 t} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f\}_{\bar{k} \in \mathbb{N}_0^d}$$

is one-to-one mapping from $S_{p,q}^t F(\mathbb{R}^d)$ onto a subspace Y of $L_p(\ell_q)$. Hence, every functional $g \in (S_{p,q}^t F(\mathbb{R}^d))'$ can be interpreted as a functional on that subspace. From the Hahn-Banach theorem we derive that g can be extended to a continuous linear functional on $L_p(\ell_q)$ with preservation of the norm. We still denote this extension by g . Now if $\varrho \in \mathcal{S}(\mathbb{R}^d)$, then Lemma 4.9 yields the existence of a sequence $\{g_{\bar{k}}\}_{\bar{k}}$ such that

$$g(\varrho) = \sum_{\bar{k} \in \mathbb{N}_0^d} \int_{\mathbb{R}^d} g_{\bar{k}}(x) 2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} \varrho](x) dx$$

and

$$\|g\| = \|g_{\bar{k}}\|_{L_{p'}(\ell_{q'})}.$$

Next we continue with a simple observations. Since $\varphi_{\bar{k}}$ is even we obtain

$$\varphi_{\bar{k}}(\xi) (\mathcal{F} \varrho)(\xi) = \varphi_{\bar{k}}(-\xi) (\mathcal{F}^{-1} \varrho)(-\xi), \quad \xi \in \mathbb{R}^d.$$

Applying the inverse Fourier transform to both sides of this identity it follows

$$\mathcal{F}^{-1}[\varphi_{\bar{k}}(\xi) (\mathcal{F} \varrho)(\xi)](x) = \mathcal{F}[\varphi_{\bar{k}}(\xi) (\mathcal{F}^{-1} \varrho)(\xi)](x).$$

We put $h_{\bar{k}} := 2^{|\bar{k}|_1 t} g_{\bar{k}}$, $\bar{k} \in \mathbb{N}_0^d$. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in $L_2(\mathbb{R}^d)$. Because of $\varphi_{\bar{k}}$ is real-valued, we find

$$\begin{aligned} \int_{\mathbb{R}^d} h_{\bar{k}}(x) \mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} \varrho](x) dx &= \langle h_{\bar{k}}, \overline{\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} \varrho]} \rangle = \langle h_{\bar{k}}, \mathcal{F}[\varphi_{\bar{k}} \mathcal{F}^{-1} \varrho] \rangle = \langle h_{\bar{k}}, \mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} \varrho] \rangle \\ &= (\mathcal{F}[\varphi_{\bar{k}} \mathcal{F}^{-1} h_{\bar{k}}])(\varrho). \end{aligned}$$

Altogether this proves the identity

$$g(\varrho) = \sum_{\bar{k} \in \mathbb{N}_0^d} (\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} h_{\bar{k}}])(\varrho)$$

for all $\varrho \in \mathcal{S}(\mathbb{R}^d)$. This means

$$g = \sum_{\bar{k} \in \mathbb{N}_0^d} \mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} h_{\bar{k}}] \quad \text{in } \mathcal{S}'(\mathbb{R}^d)$$

and $\|g\| = \|2^{-|\bar{k}|_1 t} h_{\bar{k}}\|_{L_{p'}(\ell_{q'})} < \infty$. In view of Proposition 2.6 we conclude that $g \in S_{p',q'}^{-t} F(\mathbb{R}^d)$.
Step 2. The case $1 < p < \infty$ and $q = \infty$. We proceed as in Step 1. The mapping

$$f \rightarrow \{2^{|\bar{k}|_1 t} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f\}_{\bar{k} \in \mathbb{N}_0^d}$$

is an isometric mapping from $\dot{S}_{p,\infty}^t F(\mathbb{R}^d)$ to $L_p(c_0)$. Here we use the fact that

$$\lim_{|\bar{k}|_1 \rightarrow \infty} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} f](x)| = 0 \quad \text{a.e.}$$

holds for all $f \in \dot{S}_{p,\infty}^t F(\mathbb{R}^d)$. Now the assertion is obtained from Lemma 4.9 (ii).
Step 3. Proof of (ii). We have

$$\dot{S}_{p,\min(p,q)}^t B(\mathbb{R}^d) \hookrightarrow \dot{S}_{p,q}^t F(\mathbb{R}^d) \hookrightarrow \dot{S}_{1,1}^{t-\frac{1}{p}+1} F(\mathbb{R}^d) = \dot{S}_{1,1}^{t-\frac{1}{p}+1} B(\mathbb{R}^d).$$

The known duality relations of the space $S_{p,q}^t B(\mathbb{R}^d)$, see [13] and references given there, yields

$$S_{\infty,\infty}^{-t+\frac{1}{p}-1} B(\mathbb{R}^d) \hookrightarrow (\dot{S}_{p,q}^t F(\mathbb{R}^d))' \hookrightarrow S_{\infty,\infty}^{-t+\frac{1}{p}-1} B(\mathbb{R}^d).$$

This finishes the proof. ■

4.4 Test function

Let us give some few more properties of our smooth decompositions of unity in Section 2. As a consequence of the definitions we obtain for $j \in \mathbb{N}$

$$\varphi_j(\xi) = 1 \quad \text{if} \quad \frac{3}{4} 2^j \leq \xi \leq 2^j,$$

and $\ell \in \mathbb{N}$

$$\psi_\ell(x) = 1 \quad \text{on the set} \quad \left\{x : \sup_{j=1,\dots,\ell} |x_j| \leq 2^\ell\right\} \setminus \left\{x : \sup_{j=1,\dots,\ell} |x_j| \leq \frac{3}{4} 2^\ell\right\}.$$

Example 1

Let us fix $\ell \in \mathbb{N}$ and let $\eta \in \mathcal{S}(\mathbb{R})$ such that $\text{supp } \mathcal{F}\eta \subset \{\xi \in \mathbb{R} : 0 < \xi < \frac{1}{4}\}$. We define the function g_ℓ by its Fourier transform

$$\mathcal{F}g_\ell(\xi) = \sum_{j=1}^{\ell} a_j (\mathcal{F}\eta)(\xi - \frac{7}{8} 2^j).$$

Then we arrive at

$$\mathcal{F}^{-1}(\varphi_j \mathcal{F}g_\ell)(\xi) = a_j e^{\frac{7}{8} 2^j i \xi} \eta(\xi), \quad j \leq \ell.$$

Consequently we obtain

$$\|g_\ell\|_{F_{p,q}^0(\mathbb{R})} = \|\eta\|_{L_p(\mathbb{R}^d)} \left(\sum_{j=1}^{\ell} |a_j|^q \right)^{\frac{1}{q}}.$$

Now we turn to the multi-dimensional case and introduce a new family of test functions $f_\ell : \mathbb{R}^d \rightarrow \mathbb{C}$ as follows:

$$\mathcal{F}f_\ell(x) = \theta_\ell(x_1) \cdots \theta_\ell(x_{d-1})(\mathcal{F}g_\ell)(x_d), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $\theta_1 \in \mathcal{S}(\mathbb{R})$ is a function satisfying

$$\text{supp } \theta_1 \subset \{\xi : \varphi_1(\xi) = 1\} \quad \text{and} \quad \theta_\ell(\xi) = \theta_1(2^{-\ell+1}\xi).$$

Clearly,

$$\text{supp } \theta_\ell \subset \{\xi : \varphi_\ell(\xi) = 1\} \quad \text{and} \quad \text{supp } (\mathcal{F}f) \subset \{x : \psi_\ell(x) = 1\}.$$

By means of the cross norm property we obtain

$$\begin{aligned} \|f_\ell|S_{p,q}^0 F(\mathbb{R}^d)\| &= \left(\prod_{j=1}^{d-1} \|\mathcal{F}^{-1}\theta_\ell|F_{p,q}^0(\mathbb{R})\| \right) \|g_\ell|F_{p,q}^0(\mathbb{R})\| \\ &= \left(\prod_{j=1}^{d-1} \|\mathcal{F}^{-1}\theta_\ell|L_p(\mathbb{R})\| \right) \|g_\ell|L_p(\mathbb{R})\| \\ &= 2^{(\ell-1)(1-\frac{1}{p})(d-1)} \|\mathcal{F}^{-1}\theta_1|L_p(\mathbb{R}^d)\|^{d-1} \|g_\ell|F_{p,q}^0(\mathbb{R})\| \\ &= C_1 2^{\ell(1-\frac{1}{p})(d-1)} \left(\sum_{j=1}^{\ell} |a_j|^q \right)^{\frac{1}{q}} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \|f_\ell|F_{p,q}^0(\mathbb{R}^d)\| &= \left(\prod_{j=1}^{d-1} \|\mathcal{F}^{-1}\theta_\ell|L_p(\mathbb{R})\| \right) \|g_\ell|L_p(\mathbb{R})\| \\ &= C_1 2^{\ell(1-\frac{1}{p})(d-1)} \|g_\ell|L_p(\mathbb{R})\| \end{aligned}$$

for appropriate positive constant C_1 . Using the Littlewood-Paley characterization of $L_p(\mathbb{R})$, $1 < p < \infty$, it follows

$$\|f_\ell|F_{p,q}^0(\mathbb{R}^d)\| \asymp 2^{\ell(1-\frac{1}{p})(d-1)} \left(\sum_{j=1}^{\ell} |a_j|^2 \right)^{\frac{1}{2}}, \quad \ell \in \mathbb{N}. \tag{4.7}$$

Example 2

Let us consider a function $g \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g \subset \{t \in \mathbb{R} : 3/4 \leq |t| \leq 1\}$. For $\ell \in \mathbb{N}_0$ we define

$$g_\ell(t) := g(2^{-\ell}t) \quad \text{and} \quad f_\ell(x) := \mathcal{F}^{-1}[g_\ell(\xi_1)g_0(\xi_2) \cdots g_0(\xi_d)](x).$$

Then we find

$$\begin{aligned} \|f_\ell|F_{p,q}^t(\mathbb{R}^d)\| &= 2^{t\ell} \|\mathcal{F}^{-1}[g_\ell(\xi_1)g_0(\xi_2) \cdots g_0(\xi_d)]|L_p(\mathbb{R}^d)\| \\ &= \|\mathcal{F}^{-1}[g_0(\xi_1)g_0(\xi_2) \cdots g_0(\xi_d)]|L_p(\mathbb{R}^d)\| 2^{\ell(t+1-\frac{1}{p})} \end{aligned}$$

and

$$\begin{aligned} \|f_\ell|S_{p,q}^t F(\mathbb{R}^d)\| &= 2^{t\ell} \|\mathcal{F}^{-1}[g_\ell(\xi_1)g_0(\xi_2) \cdots g_0(\xi_d)]|L_p(\mathbb{R}^d)\| \\ &= \|\mathcal{F}^{-1}[g_0(\xi_1)g_0(\xi_2) \cdots g_0(\xi_d)]|L_p(\mathbb{R}^d)\| 2^{\ell(t+1-\frac{1}{p})}. \end{aligned}$$

Example 3

We consider the same functions g_ℓ as in Example 2. This time we define

$$f_\ell(x) := \mathcal{F}^{-1}[g_\ell(\xi_1)g_\ell(\xi_2) \cdots g_\ell(\xi_d)](x).$$

Let $G_\ell(\xi) := g_\ell(\xi_1)g_\ell(\xi_2) \cdots g_\ell(\xi_d)$. As above we conclude

$$\|f_\ell|S_{p,q}^t F(\mathbb{R}^d)\| = 2^{t\ell} \|\mathcal{F}^{-1}G_\ell|L_p(\mathbb{R}^d)\| = C_3 2^{d\ell(t+1-\frac{1}{p})}, \quad \ell \in \mathbb{N},$$

and

$$\|f_\ell|F_{p,q}^t(\mathbb{R}^d)\| = 2^{t\ell} \|\mathcal{F}^{-1}G_\ell|L_p(\mathbb{R}^d)\| = C_3 2^{d\ell(\frac{t}{d}+1-\frac{1}{p})}, \quad \ell \in \mathbb{N}.$$

Example 4

Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\text{supp } \mathcal{F}g \subset [0, \frac{1}{4}]^d$. We define

$$f_\ell(x) := \sum_{j=1}^{\ell} a_j e^{i\frac{7}{8}2^j x_1} g(x), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Then we have

$$\mathcal{F}f_\ell(\xi) = \sum_{j=1}^{\ell} a_j (\mathcal{F}g)(\xi_1 - \frac{7}{8}2^j, \xi_2, \dots, \xi_d).$$

We obtain

$$\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F}f_\ell](x) = \sum_{j=1}^{\ell} \delta_{\bar{k},(j,0,\dots,0)} a_j e^{i\frac{7}{8}2^j x_1} g(x)$$

and

$$\mathcal{F}^{-1}[\psi_j \mathcal{F}f_\ell](x) = a_j e^{i\frac{7}{8}2^j x_1} g(x), \quad j \leq \ell.$$

This leads to

$$\|f_\ell|F_{p,q}^t(\mathbb{R}^d)\| = \|f_\ell|S_{p,q}^t F(\mathbb{R}^d)\| = \|g|L_p(\mathbb{R}^d)\| \left(\sum_{j=1}^{\ell} 2^{j t q} |a_j|^q \right)^{1/q}.$$

Example 5

We shall modify Example 4. This time we define the function

$$f_\ell(x) := \sum_{j=1}^{\ell} a_j e^{i\frac{7}{8}2^j(x_1+\dots+x_d)} g(x), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

As above we conclude

$$\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F}f_\ell](x) = \sum_{j=1}^{\ell} \delta_{\bar{k},(j,\dots,j)} a_j e^{i\frac{7}{8}2^j(x_1+\dots+x_d)} g(x)$$

and

$$\mathcal{F}^{-1}[\psi_j \mathcal{F}f_\ell](x) = a_j e^{i\frac{7}{8}2^j(x_1+\dots+x_d)} g(x), \quad j \leq \ell.$$

Hence, we obtain

$$\|f_\ell|F_{p,q}^t(\mathbb{R}^d)\| = \|g|L_p(\mathbb{R}^d)\| \left(\sum_{j=1}^{\ell} 2^{j tq} |a_j|^q \right)^{1/q}$$

and

$$\|f_\ell|S_{p,q}^t F(\mathbb{R}^d)\| = \|g|L_p(\mathbb{R}^d)\| \left(\sum_{j=1}^{\ell} 2^{d j tq} |a_j|^q \right)^{1/q}.$$

Example 6

This example is taken from [23, 2.3.9], see also [13]. Let $\varrho \in \mathcal{S}(\mathbb{R}^d)$ be a function such that $\text{supp } \mathcal{F}\varrho \subset \{\xi : |\xi| \leq 1\}$. We define

$$h_j(x) := \varrho(2^{-j}x), \quad x \in \mathbb{R}^d, \quad j \in \mathbb{N}.$$

For all admissible p, q, t we conclude

$$\|h_j|S_{p,q}^t F(\mathbb{R}^d)\| = \|h_j|F_{p,q}^t(\mathbb{R}^d)\| = \|h_j|L_p(\mathbb{R}^d)\| = 2^{jd/p} \|\varrho|L_p(\mathbb{R}^d)\|, \quad j \in \mathbb{N}.$$

As an immediate conclusion of this example we obtain the following result.

Lemma 4.11. *Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ and $t_0, t_1 \in \mathbb{R}$.*

- (i) *An embedding $S_{p_0, q_0}^{t_0} F(\mathbb{R}^d) \hookrightarrow F_{p_1, q_1}^{t_1}(\mathbb{R}^d)$ implies $p_0 \leq p_1$.*
- (ii) *An embedding $F_{p_0, q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_{p_1, q_1}^{t_1} F(\mathbb{R}^d)$ implies $p_0 \leq p_1$.*

4.5 Proof of results in Section 3.1

Proof of Theorem 3.1. *Step 1.* Preparations. For $\bar{k} \in \mathbb{N}_0^d$ we define

$$\square_{\bar{k}} := \{j \in \mathbb{N}_0 : \text{supp } \psi_j \cap \text{supp } \varphi_{\bar{k}} \neq \emptyset\}$$

and $j \in \mathbb{N}_0$

$$\Delta_j := \{\bar{k} \in \mathbb{N}_0^d : \text{supp } \psi_j \cap \text{supp } \varphi_{\bar{k}} \neq \emptyset\}.$$

The condition $\text{supp } \psi_j \cap \text{supp } \varphi_{\bar{k}} \neq \emptyset$ implies

$$\max_{i=1, \dots, d} k_i - 1 \leq j \leq \max_{i=1, \dots, d} k_i + 1. \quad (4.8)$$

Consequently we obtain

$$|\square_{\bar{k}}| \asymp 1, \quad \bar{k} \in \mathbb{N}_0^d \quad \text{and} \quad |\Delta_j| \asymp (1+j)^{d-1}, \quad j \in \mathbb{N}_0.$$

By definition we have

$$\psi_j(x) = \sum_{\bar{k} \in \Delta_j} \varphi_{\bar{k}}(x) \psi_j(x), \quad x \in \mathbb{R}^d. \quad (4.9)$$

Step 2. The case $t > 0$. From (4.9) we have

$$\|f|F_{p,q}^t(\mathbb{R}^d)\| = \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{\bar{k} \in \Delta_j} 2^{tj - t|\bar{k}|_1} 2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f] \right|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \quad (4.10)$$

If $q \leq 1$, then

$$\begin{aligned} \|f|F_{p,q}^t(\mathbb{R}^d)\| &\leq \left\| \left(\sum_{j=0}^{\infty} \sum_{\bar{k} \in \Delta_j} |2^{t(j-|\bar{k}|_1)} 2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_1 \left\| \left(\sum_{j=0}^{\infty} \sum_{\bar{k} \in \Delta_j} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \end{aligned} \quad (4.11)$$

The last inequality is due to

$$\sup_{j \geq 0} \sup_{\bar{k} \in \Delta_j} 2^{t(j-|\bar{k}|_1)} \leq c_1 < \infty.$$

Now we turn to $q > 1$. Using Hölder's inequality we obtain from (4.10)

$$\left| \sum_{\bar{k} \in \Delta_j} 2^{tj-t|\bar{k}|_1} 2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f] \right| \leq \left(\sum_{\bar{k} \in \Delta_j} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \left(\sum_{\bar{k} \in \Delta_j} 2^{t(j-|\bar{k}|_1)q'} \right)^{1/q'},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Because of $t > 0$ and (4.8) the second sum on the right-hand side is uniformly bounded, i.e.,

$$\sup_{j \geq 0} \left(\sum_{\bar{k} \in \Delta_j} 2^{t(j-|\bar{k}|_1)q'} \right)^{1/q'} \leq c_2 < \infty.$$

Consequently, we obtain for all q

$$\|f|F_{p,q}^t(\mathbb{R}^d)\| \leq c_3 \left\| \left(\sum_{j=0}^{\infty} \sum_{\bar{k} \in \Delta_j} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}.$$

Let $\tau := \min(1, p, q)$. Interchanging the order of summation we find

$$\begin{aligned} \|f|F_{p,q}^t(\mathbb{R}^d)\|^\tau &\leq c_3^\tau \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} \sum_{j \in \square_{\bar{k}}} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}^\tau \\ &\leq c_3^\tau \sum_{i=-1}^1 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_{j+i} \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}^\tau, \end{aligned}$$

where $j := |\bar{k}|_\infty$, see (4.8). We estimate the term with $i = 0$. The terms with $i = \pm 1$ can be treated in a similar way. Let $\{\tilde{\varphi}_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ be the system defined in the proof of Proposition 2.6. Then we have

$$\begin{aligned} &\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &= \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |\mathcal{F}^{-1} \tilde{\varphi}_{\bar{k}} \psi_j \mathcal{F} [2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \end{aligned}$$

Applying Lemma 4.6 with $M_k = \tilde{\varphi}_{\bar{k}} \psi_j$ we obtain

$$\begin{aligned} &\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_j \mathcal{F} f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_4 \sup_{\bar{k} \in \mathbb{N}_0^d} \|(\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond \cdot) |S_2^r W(\mathbb{R}^d)\| \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \end{aligned} \quad (4.12)$$

where we have chosen $r \in \mathbb{N}$ such that $r > \frac{1}{\min(p,q)} + \frac{1}{2}$. To estimate the factor $\| \dots |S_2^r W(\mathbb{R}^d)| \|$ we consider several cases. First, we assume that $\min_{i=1,\dots,d} k_i \geq 1$. Then it follows

$$(\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond x) = \tilde{\varphi}_{\bar{1}}(4x) \psi_1(2^{k_1-j+2} x_1, \dots, 2^{k_d-j+2} x_d).$$

For any $\alpha \in \mathbb{N}_0^d$, since $k_1 - j + 2 = k_1 - |k|_\infty + 2 \leq 2$, we conclude the existence of a positive constant C_α such that

$$\sup_{x \in \mathbb{R}^d} |D^\alpha(\psi_1(4 \cdot 2^{\bar{k}-|k|_\infty} \diamond x))| \leq C_\alpha < \infty.$$

Furthermore

$$\|\tilde{\varphi}_{\bar{k}}(2^{\bar{k}+\bar{1}} \diamond \cdot) |S_2^r W(\mathbb{R}^d)| \| = \|\tilde{\varphi}_{\bar{1}}(4 \cdot) |S_2^r W(\mathbb{R}^d)| \| = C_r < \infty,$$

which implies

$$\sup_{\bar{k} \in \mathbb{N}^d} \|(\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond \cdot) |S_2^r W(\mathbb{R}^d)| \| \leq c_5.$$

Now we turn to the cases $\min_{i=1,\dots,d} k_i = 0$. Let us assume that $0 = k_1 = \dots = k_m < k_{m+1} \leq \dots \leq k_d$ for some $m < d$. Recall the notation from (2.5). Then

$$\begin{aligned} (\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond x) &= \tilde{\varphi}_0(2x_1) \cdot \dots \cdot \tilde{\varphi}_0(2x_m) \cdot \tilde{\varphi}_1(4x_{m+1}) \cdot \dots \cdot \tilde{\varphi}_1(4x_d) \\ &\times \psi_1(2^{-j+2} x_1, \dots, 2^{-j+2} x_m, 2^{k_{m+1}-j+2} x_{m+1}, \dots, 2^{k_d-j+2} x_d). \end{aligned}$$

Now we proceed as above and find also in this case

$$\|(\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond \cdot) |S_2^r W(\mathbb{R}^d)| \| \leq c_6.$$

Similarly we can treat all cases caused by a different ordering of the components of \bar{k} . The case $\bar{k} = \bar{0}$, $j = 0$ can be handled in the same way. Summarizing, we get

$$\sup_{\bar{k} \in \mathbb{N}_0^d} \|(\tilde{\varphi}_{\bar{k}} \psi_j)(2^{\bar{k}+\bar{1}} \diamond \cdot) |S_2^r W(\mathbb{R}^d)| \| \leq c_7 < \infty. \quad (4.13)$$

From (4.13) and (4.12) we derive

$$\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{t|\bar{k}|_1} \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_j \mathcal{F} f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \leq c_8 \|f |S_{p,q}^t F(\mathbb{R}^d)| \|.$$

We conclude that $S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$.

Step 3. The case $t = 0$. If $0 < q \leq 1$ we can argue as in (4.11) with $t = 0$. This implies

$$\|f |F_{p,q}^0(\mathbb{R}^d)| \| \leq \left\| \left(\sum_{j=0}^{\infty} \sum_{\bar{k} \in \Delta_j} |\mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}.$$

Then we continue as in Step 2 resulting in $S_{p,q}^0 F(\mathbb{R}^d) \hookrightarrow F_{p,q}^0(\mathbb{R}^d)$ if $0 < q \leq 1$.

Now we consider the case $1 < q < 2$. Here we employ complex interpolation. For $0 < p < \infty$ and $1 < q < 2$ there exist $\Theta \in (0, 1)$, $0 < p_0 < \infty$ and $1 < p_1 < \infty$ such that

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\Theta}{1} + \frac{\Theta}{2}.$$

Proposition 4.8 yields

$$S_{p,q}^0 F(\mathbb{R}^d) = [S_{p_0,1}^0 F(\mathbb{R}^d), S_{p_1,2}^0 F(\mathbb{R}^d)]_\Theta \quad \text{and} \quad F_{p,q}^0(\mathbb{R}^d) = [F_{p_0,1}^0(\mathbb{R}^d), F_{p_1,2}^0(\mathbb{R}^d)]_\Theta.$$

Finally, the claim follows from Proposition 4.7. The proof is complete. ■

Lemma 4.12. *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Then the embedding*

$$S_{p,q}^0 F(\mathbb{R}^d) \hookrightarrow F_{p,q}^0(\mathbb{R}^d) \quad \text{implies} \quad q \leq 2.$$

Proof. *Step 1.* The case $1 < p < \infty$. We use the test function from Example 1. The embedding $S_{p,q}^0 F(\mathbb{R}^d) \hookrightarrow F_{p,q}^0(\mathbb{R}^d)$ implies the existence of a constant $c > 0$ such that

$$2^{\ell(1-\frac{1}{p})(d-1)} \left(\sum_{j=1}^{\ell} |a_j|^2 \right)^{1/2} \leq c 2^{\ell(1-\frac{1}{p})(d-1)} \left(\sum_{j=1}^{\ell} |a_j|^q \right)^{1/q}$$

holds for all $\ell \in \mathbb{N}$ and all sequences $\{a_j\}_j$, see (4.6) and (4.7). This requires $q \leq 2$.

Step 2. The case $0 < p \leq 1$. Assume that $S_{p,q}^0 F(\mathbb{R}^d) \hookrightarrow F_{p,q}^0(\mathbb{R}^d)$ with $0 < p \leq 1$ and $2 < q \leq \infty$. Then we can find a triple (p_1, q_1, Θ) such that $\Theta \in (0, 1)$, $1 < p_1 < 2 < q_1 < \infty$,

$$\frac{1}{p_1} = \frac{\Theta}{p} + \frac{1-\Theta}{2} \quad \text{and} \quad \frac{1}{q_1} = \frac{\Theta}{q} + \frac{1-\Theta}{2}.$$

Complex interpolation, see Propositions 4.7 and 4.8, yields $S_{p_1,q_1}^0 F(\mathbb{R}^d) \hookrightarrow F_{p_1,q_1}^0(\mathbb{R}^d)$. But this is in contradiction with Step 1 since $p_1 > 1$ and $q_1 > 2$. ■

Proof of Theorem 3.3. As a consequence of Theorem 3.1 and Lemma 4.12 it will be enough to consider the case $t < 0$. We assume $S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$ if $t < 0$. Observe that this implies $\mathring{S}_{p,q}^t F(\mathbb{R}^d) \hookrightarrow \mathring{F}_{p,q}^t(\mathbb{R}^d)$. Then, by duality, see Proposition 4.10, we obtain $F_{p',q'}^{-t}(\mathbb{R}^d) \hookrightarrow S_{p',q'}^{-t} F(\mathbb{R}^d)$. Using the test function from Example 5 with $a_j := \delta_{j,\ell}$ we can disprove this embedding. ■

Proof of Proposition 3.4. *Step 1.* Proof of (i). Theorem 3.1 implies $\mathring{S}_{p,q}^t F(\mathbb{R}^d) \hookrightarrow \mathring{F}_{p,q}^t(\mathbb{R}^d)$ if $t > 0$. Proposition 4.10 yields $F_{p',q'}^{-t}(\mathbb{R}^d) \hookrightarrow S_{p',q'}^{-t} F(\mathbb{R}^d)$, if $1 < p < \infty$ and $1 \leq q \leq \infty$.

Step 2. Proof of (ii). Since $-t + \frac{1}{p} - 1 < -t + d(\frac{1}{p} - 1)$ and $t < 0$ we can use $g_{\bar{k}} = e^{i\bar{k}x}$ as test functions to prove that $S_{\infty,\infty}^{-t+\frac{1}{p}-1} B(\mathbb{R}^d)$ and $B_{\infty,\infty}^{-t+d(\frac{1}{p}-1)}(\mathbb{R}^d)$ are not comparable. Here one can use

$$\|e^{i\bar{k}x} |B_{\infty,\infty}^s(\mathbb{R}^d)|\| \asymp (1 + |\bar{k}|_2)^s, \quad k \in \mathbb{N}_0^d$$

and

$$\|e^{i\bar{k}x} |S_{\infty,\infty}^s B(\mathbb{R}^d)|\| \asymp \prod_{i=1}^d (1 + |k_i|)^s, \quad k \in \mathbb{N}_0^d.$$

Then, from Proposition 4.10, we can conclude that $\mathring{S}_{p,q}^t F(\mathbb{R}^d)$ and $\mathring{F}_{p,q}^t(\mathbb{R}^d)$ are incomparable and therefore $S_{p,q}^t F(\mathbb{R}^d)$ and $F_{p,q}^t(\mathbb{R}^d)$ as well. This finishes the proof. ■

4.6 Proof of the results in Section 3.2

Proof of Theorem 3.7. The claim for the case $t = 0$ is a consequence of Theorem 3.1 and a duality argument, see Proposition 4.10. The proof in case $t > (\frac{1}{\min(p,q)} - 1)_+$ will be divided into several steps.

Step 1. We shall prove the embedding under the assumptions $0 < p < \infty$, $0 < q \leq \infty$ and $t > \frac{1}{\min(p,q)}$. Let $\tau := \min(1, p, q)$. From Step 1 in the proof of Theorem 3.1 and $\square_{\bar{k}} \asymp 1$ we obtain

$$\begin{aligned} \|f|S_{p,q}^t F(\mathbb{R}^d)\|^\tau &= \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} \left| \sum_{j \in \square_{\bar{k}}} 2^{|\bar{k}|_1 t} \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_j \mathcal{F} f \right|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}^\tau \\ &\leq \sum_{i=-1}^1 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_{j+i} \mathcal{F} f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}^\tau, \end{aligned} \tag{4.14}$$

where again $j := |\bar{k}|_\infty$. It will be enough to deal with the term for $i = 0$. The others terms can be treated similarly. Since $t > \frac{1}{\min(p,q)}$ we can write $t = a + \varepsilon$ with $a > \frac{1}{\min(p,q)}$ and $\varepsilon > 0$. We put

$$g_{\bar{k}}(x) := \mathcal{F}^{-1} [2^{(|\bar{k}|_1 - jd)\varepsilon} 2^{jtd} \psi_j \mathcal{F} f](x), \quad \bar{k} \in \mathbb{N}_0^d, \quad j = |\bar{k}|_\infty.$$

It follows

$$\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1} [\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} = \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{(|\bar{k}|_1 - jd)a} \mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} g_{\bar{k}}]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \quad (4.15)$$

Next we need the related Peetre maximal function. We define

$$P_{2^{\bar{j}+1}, a} g_{\bar{k}}(x) := \sup_{z \in \mathbb{R}^d} \frac{|g_{\bar{k}}(x - z)|}{\prod_{i=1}^d (1 + |2^{j+1} z_i|^a)}, \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N}_0^d, \quad j = |\bar{k}|_\infty,$$

compare with Proposition 4.5. A standard convolution argument, given by

$$\begin{aligned} |(\mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} g_{\bar{k}}])(x - z)| &\leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \varphi_{\bar{k}})(x - z - y)| \cdot |g_{\bar{k}}(y)| dy \\ &\leq (2\pi)^{-d/2} P_{2^{\bar{j}+1}, a} g_{\bar{k}}(x) \int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \varphi_{\bar{k}})(x - z - y)| \prod_{i=1}^d (1 + |2^{j+1} (x_i - y_i)|^a) dy, \end{aligned}$$

the elementary inequality

$$(1 + |2^{j+1} (x_i - y_i)|^a) \leq 2^a (1 + |2^{j+1} z_i|^a) (1 + |2^{j+1} (x_i - z_i - y_i)|^a),$$

$i = 1, \dots, d$, and a change of variable lead to

$$\frac{|(\mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} g_{\bar{k}})(x - z)|}{\prod_{i=1}^d (1 + |2^{j+1} z_i|^a)} \leq c_1 P_{2^{\bar{j}+1}, a} g_{\bar{k}}(x) \int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \varphi_{\bar{k}})(y)| \prod_{i=1}^d (1 + |2^{j+1} y_i|^a) dy.$$

Temporarily we assume $\min_{i=1, \dots, d} k_i \geq 1$. Then

$$\int_{\mathbb{R}^d} |(\mathcal{F}^{-1} \varphi_{\bar{k}})(y)| \prod_{i=1}^d (1 + |2^{j+1} y_i|^a) dy = \prod_{i=1}^d \int_{\mathbb{R}} |\mathcal{F}^{-1} \varphi_1(t)| (1 + 2^{j+2-k_i} |t|)^a dt$$

follows. Since $k_i \leq j = |\bar{k}|_\infty$ and $\mathcal{F}^{-1} \varphi_1 \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F}^{-1} \varphi_1(t)| (1 + 2^{j+2-k_i} |t|)^a dt &= 2^{(j-k_i)a} \int_{\mathbb{R}} |\mathcal{F}^{-1} \varphi_1(t)| (2^{k_i-j} + 4|t|)^a dt \\ &\leq c_2 2^{(j-k_i)a}. \end{aligned}$$

This estimate carries over to the situation $\min_{i=1, \dots, d} k_i = 0$ by obvious modifications. Consequently

$$2^{(|\bar{k}|_1 - jd)a} \frac{|(\mathcal{F}^{-1} [\varphi_{\bar{k}} \mathcal{F} g_{\bar{k}}])(x - z)|}{\prod_{i=1}^d (1 + 2^{j+1} |z_i|^a)} \leq c_3 P_{2^{\bar{j}+1}, a} g_{\bar{k}}(x)$$

with a constant c_3 independent of x and $\{g_{\bar{k}}\}_{\bar{k}}$. Obviously, this implies

$$\begin{aligned} 2^{(|\bar{k}|_1 - jd)a} |(\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} g_{\bar{k}}])(x)| &\leq \sup_{z \in \mathbb{R}^d} \frac{2^{(|\bar{k}|_1 - jd)a} |(\mathcal{F}^{-1}[\varphi_{\bar{k}} \mathcal{F} g_{\bar{k}}])(x - z)|}{\prod_{i=1}^d (1 + |2^{j+1} z_i|^a)} \\ &\leq c_3 P_{2^{\bar{j}+\bar{1}}, a} g_{\bar{k}}(x), \end{aligned}$$

which results in the estimate

$$\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \leq c_3 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |P_{2^{\bar{j}+\bar{1}}, a} g_{\bar{k}}(\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)},$$

see (4.15). Now, applying Proposition 4.5 with respect to $\{g_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ and with a_{k_i} chosen to be 2^{j+1} , $i = 1, \dots, d$, $j = |\bar{k}|_\infty$, we obtain

$$\begin{aligned} &\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_4 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |\mathcal{F}^{-1}[2^{(|\bar{k}|_1 - jd)\varepsilon} 2^{jtd} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_4 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} \sum_{j \in \square_{\bar{k}}} |\mathcal{F}^{-1}[2^{(|\bar{k}|_1 - jd)\varepsilon} 2^{jtd} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &= c_4 \left\| \left(\sum_{j=0}^{\infty} |2^{jtd} \mathcal{F}^{-1}[\psi_j \mathcal{F} f]|^q \sum_{\bar{k} \in \Delta_j} 2^{(|\bar{k}|_1 - jd)\varepsilon} \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \end{aligned}$$

Our assumption $\varepsilon > 0$ guarantees

$$\begin{aligned} &\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \leq c_5 \left\| \left(\sum_{j=0}^{\infty} |2^{jtd} \mathcal{F}^{-1}[\psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &= c_5 \|f\|_{F_{p,q}^{td}(\mathbb{R}^d)}, \end{aligned}$$

see (4.8). Inserting this into (4.14) and carrying out the estimates of the other terms in the same way, the claim follows.

Substep 2. We shall prove the embedding under the assumptions $1 < p < \infty$, $1 \leq q \leq \infty$ and $t > 0$. This time we use Proposition 4.3. Starting point is inequality (4.14). As above it will be enough to deal with $j := |\bar{k}|_\infty$. The remaining terms can be treated in a similar way. Applying Proposition 4.3 in connection with a decomposition argument as in (2.6) we obtain

$$\begin{aligned} &\left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\bar{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &= \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |\mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} [2^{|\bar{k}|_1 t} \mathcal{F}^{-1} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_6 \left\| \left(\sum_{\bar{k} \in \mathbb{N}_0^d} |2^{|\bar{k}|_1 t} \mathcal{F}^{-1}[\psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c_6 \left\| \left(\sum_{j=0}^{\infty} |2^{jtd} \mathcal{F}^{-1}[\psi_j \mathcal{F} f]|^q \sum_{\bar{k} \in \Delta_j} 2^{(|\bar{k}|_1 - jd) tq} \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}. \end{aligned}$$

Because of $t > 0$ we conclude that

$$\left\| \left(\sum_{\vec{k} \in \mathbb{N}_0^d} |2^{|\vec{k}|_1 t} \mathcal{F}^{-1}[\varphi_{\vec{k}} \psi_j \mathcal{F} f]|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} \leq c_7 \|f\|_{F_{p,q}^{td}(\mathbb{R}^d)}.$$

From this the claim follows.

Step 3. Let $0 < p, q < \infty$ and $t > (\frac{1}{\min(p,q)} - 1)_+$. We shall proceed by interpolation.

Substep 3.1. Assume that $\min(p, q) \leq 1$ and $p \leq q$. Since $t > \frac{1}{p} - 1$ we choose $p_0 > 1$, $0 < \Theta < 1$ and $\varepsilon > 0$ such that

$$t = \varepsilon + \frac{1}{p} - \frac{1}{p_0} + \frac{\Theta}{p_0}.$$

Next we define (p_0, q_0) , (p_1, q_1) by

$$\frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}.$$

Now we put $t_0 := \varepsilon$ and $t_1 := \frac{1}{\min(p_1, q_1)} + \varepsilon = \frac{1}{p_1} + \varepsilon$ since $p_1 \leq q_1$. Hence we obtain

$$t = (1 - \Theta)t_0 + \Theta t_1 \quad \text{and} \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}.$$

Proposition 4.8 yields

$$F_{p,q}^{td}(\mathbb{R}^d) = [F_{p_0,q_0}^{t_0 d}(\mathbb{R}^d), F_{p_1,q_1}^{t_1 d}(\mathbb{R}^d)]_{\Theta} \quad \text{and} \quad S_{p,q}^t F(\mathbb{R}^d) = [S_{p_0,q_0}^{t_0} F(\mathbb{R}^d), S_{p_1,q_1}^{t_1} F(\mathbb{R}^d)]_{\Theta}.$$

In view of Proposition 4.7, Steps 1 and 2 we find $F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$.

Substep 3.2. Assume that $\min(p, q) \leq 1$ and $q < p$. It is enough to interchange the roles of p and q in Substep 3.1. ■

Remark 4.13. The interpolation argument in Substep 3.1 does not extend to the case $q_0 = q_1 = \infty$. It is known that

$$[F_{p_0,\infty}^{t_0 d}(\mathbb{R}^d), F_{p_1,\infty}^{t_1 d}(\mathbb{R}^d)]_{\Theta} \neq F_{p,\infty}^{td}(\mathbb{R}^d)$$

if $F_{p_0,\infty}^{t_0 d}(\mathbb{R}^d) \neq F_{p_1,\infty}^{t_1 d}(\mathbb{R}^d)$, see [32]. However, one could apply the \pm method of Gustavsson and Peetre, denoted by $\langle \cdot, \cdot, \Theta \rangle$, to obtain

$$\langle F_{p_0,\infty}^{t_0 d}(\mathbb{R}^d), F_{p_1,\infty}^{t_1 d}(\mathbb{R}^d), \Theta \rangle = F_{p,\infty}^{td}(\mathbb{R}^d),$$

see [32]. However, there is no proof of the assertion

$$\langle S_{p_0,\infty}^{t_0} F(\mathbb{R}^d), S_{p_1,\infty}^{t_1} F(\mathbb{R}^d), \Theta \rangle = S_{p,\infty}^t F(\mathbb{R}^d)$$

available in the literature.

Proof of Theorem 3.9. By Theorem 3.2 and Lemma 4.12 it will be enough to deal with $t < 0$. We assume that $F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$ if $t < 0$. This implies $\mathring{F}_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow \mathring{S}_{p,q}^t F(\mathbb{R}^d)$ and therefore, by duality, $S_{p',q'}^{-t} F(\mathbb{R}^d) \hookrightarrow F_{p',q'}^{-td}(\mathbb{R}^d)$. Applying Example 4 with $a_j := \delta_{j,\ell}$ we come to a contradiction. ■

Proof of Proposition 3.10. *Step 1.* Proof of (i). Since $0 < p < 1$ we know

$$[\dot{S}_{p,q}^t F(\mathbb{R}^d)]' = S_{\infty,\infty}^{-t+\frac{1}{p}-1} B(\mathbb{R}^d) \quad \text{and} \quad [\dot{F}_{p,q}^{td}(\mathbb{R}^d)]' = B_{\infty,\infty}^{-td+d(\frac{1}{p}-1)}(\mathbb{R}^d),$$

see Proposition 4.10. Assuming $F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$ we get $\dot{F}_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow \dot{S}_{p,q}^t F(\mathbb{R}^d)$ and therefore $S_{\infty,\infty}^{-t+\frac{1}{p}-1} F(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^{-td+d(\frac{1}{p}-1)}(\mathbb{R}^d)$. Since $-td+d(\frac{1}{p}-1) > -t+\frac{1}{p}-1 \geq 0$ this is impossible (again it will be enough to use $e^{i\bar{k}x}$ as test functions). Hence $F_{p,q}^{td}(\mathbb{R}^d) \not\hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$. By employing the test function in Example 4 with $a_j := \delta_{j,\ell}$ we can disprove the embedding $S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^{td}(\mathbb{R}^d)$. *Step 2.* Proof of (ii). We argue as in the proof of Theorem 3.1 replacing $F_{p,q}^t(\mathbb{R}^d)$ by $F_{p,q}^{td}(\mathbb{R}^d)$ and taking into account that $t < 0$. The proof is complete. \blacksquare

4.7 Proofs of the optimality assertions

Let us recall some results about embeddings of Lizorkin-Triebel spaces.

Lemma 4.14. *Let $0 < p < p_0 < \infty$ and $0 < q, q_0 \leq \infty$.*

- (i) *The embedding $F_{p,q}^t(\mathbb{R}^d) \hookrightarrow F_{p_0,q_0}^{t_0}(\mathbb{R}^d)$ holds if and only if $t_0 - \frac{d}{p_0} \leq t - \frac{d}{p}$.*
- (ii) *The embedding $S_{p,q}^t F(\mathbb{R}^d) \hookrightarrow S_{p_0,q_0}^{t_0} F(\mathbb{R}^d)$ holds if and only if $t_0 - \frac{1}{p_0} \leq t - \frac{1}{p}$.*

Note that in case $p = p_0$ and $t = t_0$, that is the embedding $F_{p,q}^t(\mathbb{R}^d) \hookrightarrow F_{p,q_0}^t(\mathbb{R}^d)$, holds true if and only if $q \leq q_0$. A similar statement is true for Lizorkin-Triebel spaces of dominating mixed smoothness. The assertion (i) in Lemma 4.14 can be found in [9], [23, 2.7.1] (sufficiency) and in [18] (necessity). In case of Triebel-Lizorkin spaces of dominating mixed smoothness we refer to [17] and [8] (sufficiency). Necessity can be traced back to the isotropic case by standard arguments (one considers tensor products of appropriate test functions). Now we are in position to prove the optimality assertions.

Proof of Theorem 3.5. Assuming $S_{p_0,q_0}^{t_0} F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$, Lemma 4.11 yields $p_0 \leq p$. Applying Example 2 we derive

$$t + 1 - \frac{1}{p} \leq t_0 + 1 - \frac{1}{p_0} \iff t - \frac{1}{p} \leq t_0 - \frac{1}{p_0}.$$

If $p_0 = p$ and $t_0 = t$ we use in addition Example 4. With $a_j := 2^{-jt}$, the embedding $S_{p,q_0}^t F(\mathbb{R}^d) \hookrightarrow F_{p,q}^t(\mathbb{R}^d)$ implies that $q_0 \leq q$. Altogether Lemma 4.14 implies that $S_{p_0,q_0}^{t_0} F(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$. \blacksquare

Proof of Theorem 3.11. Assuming $F_{p_0,q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$ Lemma 4.11 implies $p_0 \leq p$. Next we employ Example 3. Then the embedding $F_{p_0,q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_{p,q}^t F(\mathbb{R}^d)$ yields

$$d\left(\frac{t_0}{d} + 1 - \frac{1}{p_0}\right) \geq d\left(t + 1 - \frac{1}{p}\right) \iff t_0 - \frac{d}{p_0} \geq dt - \frac{d}{p}.$$

In case $p_0 = p$ and $t_0 = td$ we use Example 5, again with $a_j := 2^{-jt}$ to obtain $q_0 \leq q$. As a consequence of Lemma 4.14 we arrive at $F_{p_0,q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow F_{p,q}^{td}(\mathbb{R}^d)$. \blacksquare

Proof of Theorem 3.12. Assuming $F_{p,q}^{td}(\mathbb{R}^d) \hookrightarrow S_{p_0,q_0}^{t_0}F(\mathbb{R}^d)$ Lemma 4.11 implies $p \leq p_0$. Next we apply Example 3 and get

$$d\left(t_0 + 1 - \frac{1}{p_0}\right) \leq d\left(t + 1 - \frac{1}{p}\right) \iff t_0 - \frac{1}{p_0} \leq t - \frac{1}{p}.$$

Working with Example 4 with $a_j := 2^{-jt}$ we obtain $q \leq q_0$ in case $p = p_0$ and $t = t_0$. In a view of Lemma 4.14 we conclude that $S_{p,q}^tF(\mathbb{R}^d) \hookrightarrow S_{p_0,q_0}^{t_0}F(\mathbb{R}^d)$. ■

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